Constructing an Adams Operation on BP⟨n⟩

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Introduction

- Objects: $\mathrm{MU}_{(p)}$, E_n, $\mathrm{BP}\langle n\rangle$
- Ingredients: Adams operation on $\mathrm{MU}_{(p)}$ (stable Adams conjecture), Adams operation on E_n (Hopkins–Miller)
- Goal: Constructing a locally unipotent Adams operation on $\mathrm{BP}\langle n\rangle$ that is compatible with the Adams operations above

Theorem (5.4, Burkland–Hahn–Levy–Schlank)

The $(\mathbb{E}_1 \otimes \mathbb{A}_2)$*-*$MU_{(p)}$*-algebra underlying the* $\mathbb{E}_3$*-*$MU_{(p)}$*-algebra* $\mathrm{BP}\langle \mathrm{n} \rangle$ *admits a lift*

$$\mathrm{BP}\langle \mathrm{n} \rangle^{\Psi} \in \mathrm{Alg}_{\mathbb{E}_1 \otimes \mathbb{A}_2}(\mathrm{Mod}(\mathrm{Sp}^{B\,\mathbb{Z}}; MU^{\Psi}_{(p)}))$$

such that

- *there is a map* $\iota : \mathrm{BP}\langle \mathrm{n} \rangle^{\Psi} \to E_n^{\Psi}$ *in* $\mathrm{Alg}_{\mathbb{E}_1}(\mathrm{Mod}(\mathrm{Sp}^{B\,\mathbb{Z}}; MU^{\Psi}_{(p)}))$.

Theorem (BHLS (cont.))

- *an identification in* $\mathrm{Alg}_{\mathbb{E}_1}(\mathrm{Sp}^{B\mathbb{Z}})$*:*

$$\begin{tikzcd}
L_{T(n)}\,\mathrm{BP}\langle \mathrm{n}\rangle^{\Psi} \arrow[r, "\iota"] \arrow[d, "\cong"'] & E_n^{\Psi} \arrow[d, "\cong"] \\
(E_n^{\Psi})^{h\mu_{p^n-1} \rtimes \mathrm{Gal}(\mathbb{F}_p)} \arrow[r] & E_n^{\Psi}
\end{tikzcd}$$

where $\mu_{p^n-1} \rtimes \mathrm{Gal}(\mathbb{F}_p)$ *fits into*

$$\begin{tikzcd}
\mu_{p^n-1} \arrow[r, hook] \arrow[d, hook] & \mu_{p^n-1} \rtimes \mathrm{Gal}(\mathbb{F}_p) \arrow[r] \arrow[d, hook] & \mathrm{Gal}(\mathbb{F}_p) \arrow[d, "\cong"] \\
\mathbb{S}_n \arrow[r, hook] & \mathbb{G}_n \arrow[r] & \mathrm{Gal}(\mathbb{F}_p)
\end{tikzcd}$$

- *the underlying* $\mathbb{Z}$*-action on* $\mathrm{BP}\langle \mathrm{n}\rangle$ *is locally unipotent in p-complete spectra after p-completion.*

Outline of the construction

- Refine the $\mathbb{E}_3$-algebra structure on E_n to an $\mathbb{E}_3$-$\mathrm{MU}_{(p)}$-algebra together with a $\mathbb{E}_3$-$\mathrm{MU}_{(p)}$-algebra map:

  $$\iota : \mathrm{BP}\langle \mathrm{n} \rangle \to E_n.$$

  (Theorem 5.9, Theorem 5.10)
- Use the self-centrality of E_n and universal property of center to show that the $\mathbb{E}_3$-$\mathrm{MU}_{(p)}$-algebra structure on E_n corresponds to an $\mathbb{E}_4$-algebra map $\mathrm{MU}_{(p)} \to E_n$. (Theorem 5.12, 5.13)
- Base case: Show that the underlying $\mathbb{E}_3$-algebra map of the $\mathbb{E}_4$-algebra map $\mathrm{MU}_{(p)} \to E_n$ can be refined to a $\mathbb{Z}$-equivariant $\mathbb{E}_3$-algebra map $\mathrm{MU}_{(p)} \to E_n$, so E_n^{Ψ} lifts to a $\mathbb{Z}$-equivariant $\mathbb{E}_2$-$\mathrm{MU}_{(p)}{}^{\Psi}$-algebra. (Theorem 5.16)

- Induction: Refine the $\mathbb{E}_1$-$\mathrm{MU_{(p)}}$-algebra of $\mathrm{BP}\langle\mathrm{n}\rangle$ to

$$\mathrm{BP}\langle\mathrm{n}\rangle^{\Psi} \in \mathrm{Alg}_{\mathbb{E}_1}(\mathrm{Mod}(\mathrm{Sp}^{B\mathbb{Z}}; \mathrm{MU_{(p)}}^{\Psi})).$$

  and extend $\iota : \mathrm{BP}\langle\mathrm{n}\rangle \to E_n$ to a $\mathbb{Z}$-equivariant map between $\mathbb{E}_1$-$\mathrm{MU_{(p)}}^{\Psi}$-algebras. (Theorem 5.29)
- Lift the underlying $\mathbb{E}_1 \otimes \mathbb{A}_2$-$\mathrm{MU_{(p)}}$-algebra of $\mathrm{BP}\langle\mathrm{n}\rangle$ to

$$\mathrm{BP}\langle\mathrm{n}\rangle^{\Psi} \in \mathrm{Alg}_{\mathbb{E}_1 \otimes \mathbb{A}_2}(\mathrm{Mod}(\mathrm{Sp}^{B\mathbb{Z}}; \mathrm{MU_{(p)}}^{\Psi})).$$

  (Theorem 5.33)
- Use the fact that $\Phi^l \in \mathbb{G}_n$ is central to show that the Adams operation and taking the homotopy fixed point commute + prove local indempotence.

Adams operation on $\mathrm{MU}_{(\mathrm{p})}$ and E_n

Construction (Adams operation on E_n)

Let $l = 3$ for $p = 2$ and $l = 2$ for odd p for this talk. Let $\Psi^l \in \mathbb{G}_n : E_n \to E_n$ that arises from the $\mathbb{Z}_p^\times$-action on the chosen formal group of height n over $\overline{\mathbb{F}_p}$, such that it acts on $\pi_{2k}(E_n)$ by l^k.

Construction (Adams operation on $\mathrm{MU}_{(\mathrm{p})}$)

The stable Adams conjecture gives us

$$\begin{tikzcd}
BU_{(p)} \arrow[rr, "\Psi^l"] \arrow[dr, "J"'] & & BU_{(p)} \arrow[dl, "J"] \\
& BSL_1(\mathbb{S}_{(p)}) &
\end{tikzcd}$$

Taking the Thom spectrum, we get an Admas operation on $\mathrm{MU}_{(\mathrm{p})}$.

Notation

- The category $\mathrm{Mod}(\mathrm{MU_{(p)}}^{\Psi}; \mathrm{Sp}^{B\mathbb{Z}})$ has objects $\{(X, \Psi_X) : X \in \mathrm{Sp}^{B\mathbb{Z}} \cap \mathrm{Mod}(MU), \mathrm{MU_{(p)}}\text{-module map } \Psi_X : X \to \Psi_*(X)\}$ and morphisms $f : X \to Y$ are commutative diagrams of $\mathrm{MU_{(p)}}$-module maps

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\Psi_X"'] & Y \arrow[d, "\Psi_Y"] \\
\Psi_* X \arrow[r, "\Psi_* f"] & \Psi_* Y
\end{tikzcd}$$

- $\mathbb{S}_{(p)}(j)$: invertible object in p-complete spectra where Ψ^l acts by multiplying by l^j on $\pi_0(\mathbb{S}_{(p)})^{\times}$.
- $\mathrm{MU_{(p)}}(j) = \mathrm{MU_{(p)}} \otimes_{\mathbb{S}_p} \mathbb{S}_p(j)$.
- $S^{n,[l]}$: sphere with a degree-l self map.

$\mathbb{E}_3$-$\mathrm{MU}_{(p)}$-algebra structure on E_n

We have a classical result that there is an isomorphism

$$L_{T(n)}\,\mathrm{BP}\langle n\rangle \xrightarrow{\cong} (E_n)^{h\mu_{p^n-1}\rtimes \mathrm{Gal}(\mathbb{F}_p)},$$

and our goal here is to define a $\mathrm{MU}_{(p)}$-module structure on E_n. In the construction of $\mathrm{BP}\langle n\rangle$, we define $\mathrm{MU}_{(p)}[y]$ as the Thom spectrum of

$$\mathbb{N} \xrightarrow{p^n-1} \mathbb{Z} \to \mathrm{Pic}(\mathrm{MU}_{(p)}).$$

Analogously, we define $\mathrm{MU}_{(p)}[y^{1/(p^n-1)}]$ as as the Thom spectrum of

$$\mathbb{N} \xrightarrow{1} \mathbb{Z} \to \mathrm{Pic}(\mathrm{MU}_{(p)}).$$

We define

$$\mathrm{BP}\langle \mathrm{n}\rangle[v_n^{1/(p^n-1)}] := \mathrm{MU}_{(\mathrm{p})}[y^{1/(p^n-1)}] \otimes_{\mathrm{MU}_{(\mathrm{p})}} \mathrm{BP}\langle \mathrm{n}\rangle$$

and we obtained a $\mathbb{E}_3$-$\mathrm{MU}_{(\mathrm{p})}[y^{1/(p^n-1)}]$-algebra.

Lemma

The commutative algebra map

$$\mathbb{W}(\mathbb{F}_p) \otimes \mathrm{MU}_{(\mathrm{p})}[y^{\pm 1}] \to \mathrm{colim}_k\, \mathbb{W}(\mathbb{F}_{p^k}) \otimes \mathrm{MU}_{(\mathrm{p})}[y^{\pm 1/(p^n-1)}]$$

is a $\mu_{p^n-1} \rtimes \mathrm{Gal}(\mathbb{F}_p)$ *pro-Galois extension.*

Proof. (sketch)
Lifting the $\mu_{p^n-1} \rtimes \mathrm{Gal}(\mathbb{F}_p)$ pro-Galois extension on homotopy groups.

Proposition

There is an identification of underlying $\mathbb{E}_3$-algebra

$$\begin{tikzcd}
L_{T(n)} \mathrm{BP}\langle \mathrm{n} \rangle \arrow[r] \arrow[d, "\cong"'] & L_{T(n)}(\operatorname{colim}_k \mathbb{W}(\mathbb{F}_{p^k}) \otimes \mathrm{BP}\langle \mathrm{n} \rangle [v_n^{1/(p^n-1)}]) \arrow[d, "\cong"] \\
(E_n)^{h\mu_{p^n-1} \rtimes \mathrm{Gal}(\mathbb{F}_p)} \arrow[r] & E_n
\end{tikzcd}$$

Proof. (sketch)

The homotopy group of $L_{T(n)}(\operatorname{colim}_k \mathbb{W}(\mathbb{F}_{p^k}) \otimes \mathrm{BP}\langle \mathrm{n} \rangle [v_n^{1/(p^n-1)}])$ agrees with a Lubin–Tate theory, so it agrees as an $\mathbb{E}_3$-algebra with a Lubin–Tate theory, which is E_n as there is one formal group over an algebraically closed field up to isomorphism.

Therefore, we obtain a refinement of the underlying $\mathbb{E}_3$-algebra structure on E_n to an $\mathbb{E}_3$-$\mathrm{MU}_{(p)}$-algebra structure and a map of $\mathbb{E}_3$-$\mathrm{MU}_{(p)}$-algebra:

$$\iota : \mathrm{BP}\langle \mathrm{n} \rangle \to E_n.$$

The $\mathbb{Z}$-equivariant $\mathbb{E}_2$-$\mathrm{MU}_{(p)}$-algebra structure on E_n

It's a fact [Lur17, Section 5.3] that for for an $\mathbb{E}_m$-algebra R, its center $\mathcal{Z}_{\mathbb{E}_m}(R)$ is the terminal $\mathbb{E}_{m+1}$-algebra A such that R lifts to a $\mathbb{E}_m$-A-algebra. Explicitly, we have that R lifts to a $\mathbb{E}_m$-$\mathcal{Z}_{\mathbb{E}_m}$-algebra, and for any $\mathbb{E}_{m+1}$-algebra B such that R lifts to a $\mathbb{E}_m$-B-algebra, there is a corresponding $\mathbb{E}_{m+1}$-algebra map $B \to \mathcal{Z}_{\mathbb{E}_m} R$.

Proposition

The $\mathbb{E}_m$ center of E_n is isomorphic to E_n for $m \geq 2$.

Therefore, the structure map of $\mathbb{E}_3$-$\mathrm{MU}_{(p)}$-algebra:

$$f : \mathrm{MU}_{(p)} \to E_n$$

corrsponds to a map of $\mathbb{E}_4$-algebra:

$$\bar{f} : \mathrm{MU}_{(p)} \to \mathcal{Z}_{\mathbb{E}_3} E_n \cong E_n.$$

Our goal is to refine E_n^{Ψ} to a $\mathbb{Z}$-equivariant $\mathbb{E}_2$-${\mathrm{MU}_{(p)}}^{\Psi}$-algebra from the $\mathbb{E}_3$-$\mathrm{MU}_{(p)}$-algebra structure on E_n, so it suffices to refine the $\mathbb{E}_4$-algebra map $\bar{\iota}$ to a $\mathbb{Z}$-equivariant $\mathbb{E}_3$-algebra map

$${\mathrm{MU}_{(p)}}^{\Psi} \to E_n^{\Psi}.$$

Proposition

The underlying $\mathbb{E}_3$*-algebra map of any* $\mathbb{E}_4$*-algebra map* $\bar{f}$ *can be refined to a* $\mathbb{Z}$*-equivariant* $\mathbb{E}_3$*-algebra map*

$$\mathrm{MU_{(p)}}^{\Psi} \to E_n^{\Psi}.$$

Proof. (sketch)
By [ACB19, Theorem 3.5], the space of $\mathbb{E}_4$-algebra map $\bar{f}$ is isomorphic to the space of nullhomotopies of the composite

$$BU(p) \xrightarrow{J} BSL(\mathbb{S}_{(p)}) \to BSL_1(E_n)$$

in the category of 4-fold loop maps, which is then equivalent to this space of nullhomotopies in pointed spaces of the composite

$$B^4BU(p) \xrightarrow{J} B^5SL(\mathbb{S}_{(p)}) \to B^5SL_1(E_n).$$

We prove this theorem by showing that every nullhomotopies of this composite can be refined to a $\mathbb{Z}$-equivariant nullhomotopy of the composite

$$B^{4,[l^2]}BU(p) \xrightarrow{J} B^{5,[l^2]}SL(\mathbb{S}_{(p)}) \to B^{5,[l^2]}SL_1(E_n).$$

Applying $\Omega^{4,[l^2]}$ to the sequence above, we get a a $\mathbb{Z}$-equivariant nullhomotopy of the composite in the category of 3-fold loop maps as $S^{4,[l^2]} = S^{3,[0]} \wedge S^{1,[l^2]}$.

Corollary

The $\mathbb{E}_3$-$\mathrm{MU}_{(p)}$*-algebra structure on* E_n *can be used to refine* E_n^{Ψ} *to a* $\mathbb{Z}$*-equivariant* $\mathbb{E}_2$-${\mathrm{MU}_{(p)}}^{\Psi}$*-algebra.*

$\mathbb{E}_1$-Cell Theory

Definition (Connective algebra)

For a stable, presentably symmeytric moinodal category $\mathcal{C}$, an algebra object $A \in \mathrm{Alg}(\mathcal{C})$ is called connected if A is connective and $\pi_0(A) = \pi_0(\mathbb{1})$. Let $\mathrm{Alg}(C)^{\geq 1}$ denote the connected algebra object in $\mathcal{C}$.

Definition (Functor $\mathbb{H}$)

We define $\mathbb{H}$ to be the composite of

$$\mathrm{Alg}(C)^{\geq 1} \xrightarrow{\otimes \pi_0 \mathbb{1}} \mathrm{Alg}(\mathrm{Mod}(\mathcal{C}; \pi_0(\mathbb{1}))) \xrightarrow{\Sigma^\infty} \mathrm{Mod}(\mathcal{C}; \pi_0(\mathbb{1})).$$

Concretely, we have

$$\mathbb{H}(A) = \mathrm{fib}(A \otimes \pi_0(\mathbb{1}) \to \pi_0(\mathbb{1}) \otimes_{\pi_0(A)} \pi_0(\mathbb{1})).$$

Example

- $\mathbb{H}(\mathbb{1}\{X\}) \cong X$.
- Take $\mathcal{C} = \mathrm{Mod}(\mathrm{MU}_{(\mathrm{p})})$, then $\pi_0(\mathbb{1}) = \pi_0(\mathrm{MU}_{(\mathrm{p})}) = \mathbb{Z}_{(p)}$. It is calculated that $\pi_*(\mathbb{H}(\mathrm{BP}\langle \mathrm{n} \rangle))$ is torsion-free, finitely generated, and concentrated in positive odd degrees.
- $\mathrm{fib}(\mathbb{H}(A \coprod B) \to \mathbb{H}(A \otimes B)) \cong \mathbb{H}(A) \otimes_{\pi_0(\mathbb{1})} \mathbb{H}(B)$.

Lemma ($\mathbb{E}_1$-celullar approximation)

Let $f : A \to B$ *be a map in* $\mathrm{Alg}(\mathcal{C})^{\geq 1}$ *and let*

$$\mathbb{H}(A) = M_0 \xrightarrow{m_1} M_1 \xrightarrow{m_1} M_2 \to \cdots \to M_\infty = \mathbb{H}(B)$$

be an increasing filtration such that

1. m_i *is* k_i*-connnective for a non-decreasing sequence* k_i *with* $k_1 \geq 0$*,*
2. *for each* $i \geq 1$*, there are objects* $X_i \in \mathcal{C}_{\geq k}$ *such that* $X_i \otimes \pi_0(\mathbb{1}) \cong \mathrm{fib}(m_i)$ *and* $[X_i, Y] = 0$ *for any* $(k_i + 2)$*-connective* Y*.*

Lemma ($\mathbb{E}_1$-celullar approximation, cont.)

Then there exists a filtration in $\mathrm{Alg}(\mathcal{C})^{\geq 1}$*:*

$$A := R_0 \xrightarrow{r_1} R_1 \xrightarrow{r_2} \to \cdots \to R_\infty \to B$$

such that

1. *the map* $R_i \to B$ *is* k_{i+1}*-connective,*
2. *the functor* $\mathbb{H}$ *sends* $\{R_i\}$ *to* $\{M_i\}$*,*
3. *each* R_i *is built as a pushout*

$$\begin{tikzcd}
\mathbb{1}\{X_i\} \arrow[r, "aug"] \arrow[d] & \mathbb{1} \arrow[d] \\
R_{i-1} \arrow[r, "r_i"] & R_i
\end{tikzcd}$$

4. *the map* $R_\infty \to B$ *is* ∞*-connective.*

We apply this lemma to the category of $\mathsf{MU}_{(p)}$-module spectra. Using above examples, we will see shortly that it's relatively easy to construct a filtration on the homology level $\mathbb{H}$ (as a split Posnikov tower where we know what the fibers are) that satisfies the connective and coconnective hypothesis. We want to obtain a filtration on the spectrum level, which allows us to extend the $\mathbb{Z}$-equivariant structure level by level from $\mathsf{MU}_{(p)}$ to $\mathsf{BP}\langle n\rangle$ via taking pushout.

$\mathrm{BP}\langle \mathrm{n}\rangle$ as a $\mathbb{Z}$-equivariant $\mathbb{E}_1$-$\mathrm{MU}_{(\mathrm{p})}{}^{\Psi}$-algebra

Proposition

The underlying $\mathbb{E}_1$-$\mathrm{MU}_{(\mathrm{p})}$-*algebra of* $\mathrm{BP}\langle \mathrm{n}\rangle$ *lifts to*

$$\mathrm{BP}\langle \mathrm{n}\rangle^{\Psi} \in \mathrm{Alg}(\mathrm{Mod}(\mathrm{Sp}^{B\mathbb{Z}}; \mathrm{MU}_{(\mathrm{p})}{}^{\Psi})).$$

and $\iota : \mathrm{BP}\langle \mathrm{n}\rangle \to E_n$ *respects the* $\mathbb{Z}$-*equivariant* $\mathbb{E}_1$-$\mathrm{MU}_{(\mathrm{p})}$-*algebra structure.*

Proof.

Using $\mathbb{E}_1$-cell theory, we extend the structure map $\mathrm{MU}_{(p)} \to (\Psi^l_{\mathrm{MU}_{(p)}\,*})\,\mathrm{BP}\langle n\rangle$ by a filtration along the structure map $\mathrm{MU}_{(p)} \to \mathrm{BP}\langle n\rangle$ to obtain the dashed map.

$$\begin{tikzcd}
 & \mathrm{BP}\langle n\rangle \arrow[rr, "\iota"] \arrow[dl, dashed, "\Psi_{\mathrm{BP}\langle n\rangle}"'] & & E_n \arrow[dr] & \\
(\Psi^l_{\mathrm{MU}_{(p)}})_*\,\mathrm{BP}\langle n\rangle \arrow[rrrr, bend right, "(\Psi^l_{\mathrm{MU}_{(p)}})_*\iota"'] & & \mathrm{MU}_{(p)} \arrow[ul, "g"] \arrow[ur, "f"'] \arrow[ll, dashed] \arrow[rr] & & (\Psi^l_{\mathrm{MU}_{(p)}})_* E_n
\end{tikzcd}$$

We have a filtration

$$\mathbb{H}(\mathrm{MU}_{(p)}) = 0 \to \Sigma^1(\pi_1(\mathbb{H}(\mathrm{BP}\langle n\rangle)) \to \Sigma^1(\pi_1(\mathbb{H}(\mathrm{BP}\langle n\rangle))\oplus\Sigma^3(\pi_3(\mathbb{H}(\mathrm{BP}\langle n\rangle))$$

$$\bigoplus_{i>0}\Sigma^{2i-1}(\pi_{2i-1}(\mathbb{H}(\mathrm{BP}\langle n\rangle)) \cong \mathbb{H}(\mathrm{BP}\langle n\rangle)$$

where each fiber is $\pi_{2i-1}(\mathbb{H}(\mathrm{BP}\langle n\rangle))$ which, by the examples above, is a finite sum of copies of $\mathbb{Z}_{(p)}$.

We take X_i to be a sum of copies of $\Sigma^{2i-2}\,\mathrm{MU}_{(\mathrm{p})}$, so we have a filtration

$$\mathrm{MU}_{(\mathrm{p})} = R_0 \xrightarrow{r_1} R_1 \xrightarrow{r_2} \to \cdots \to R_\infty \xrightarrow{\cong} \mathrm{BP}\langle \mathrm{n} \rangle$$

as $\mathbb{E}_1$-$\mathrm{MU}_{(\mathrm{p})}$-algebra. Here R_i is obtained by R_{i-1} via the lower pushout square

$$\begin{tikzcd}
\bigoplus \Sigma^{2i-2}\,\mathrm{MU}_{(\mathrm{p})} \arrow[r] \arrow[d] & 0 \arrow[d] \arrow[dddr, dashed, "h_i"] & & \\
\mathrm{MU}_{(\mathrm{p})}\{\bigoplus \Sigma^{2i-2}\,\mathrm{MU}_{(\mathrm{p})}\} \arrow[r, "aug"] \arrow[d] & \mathrm{MU}_{(\mathrm{p})} \arrow[d] \arrow[ddr, dashed, "g_i"] & & \\
R_{i-1} \arrow[r, "r_i"] \arrow[dr, bend right, "f_{i-1}"'] & R_i \arrow[r] \arrow[dr, dashed, "f_i"] & \mathrm{BP}\langle \mathrm{n} \rangle \arrow[r] & E_n \arrow[d] \\
& & (\Psi^l)_*(\mathrm{BP}\langle \mathrm{n} \rangle) \arrow[r] & (\Psi^l)_*(E_n)
\end{tikzcd}$$

Given the commutative diagram with all solid arrows, we want to find a lift f_i. By the universal property of pushout, it suffices to find a lift g_i. Again, by the universal property of free object, it boils down to find a lift h_i. After simplifying the diagram, we see that we need to find a dashed lift, which in this case amounts to find a nullhomotopy of $\bigoplus \Sigma^{2i-2} \mathsf{MU}_{(\mathsf{p})} \to \mathrm{hfib}((\Psi^l)_* \iota)$.

$$\begin{tikzcd}
 & \mathrm{hfib}((\Psi^l)_* \iota) =: F \arrow[d] \\
\bigoplus \Sigma^{2i-2} \mathsf{MU}_{(\mathsf{p})} \arrow[r] \arrow[d] \arrow[ur, dashed] & (\Psi^l)_*(\mathsf{BP}\langle \mathsf{n} \rangle) \arrow[d] \\
0 \arrow[r] \arrow[ur, dashed, "h_i"] & (\Psi^l)_*(E_n)
\end{tikzcd}$$

There exists a long exact sequence of homotopy group:

$$\cdots \to \pi_{2i-1}(E_n) \to \pi_{2i-2}(F) \to \pi_{2i-2}(\mathsf{BP}\langle \mathsf{n} \rangle) \hookrightarrow \pi_{2i-2}(E_n) \to \cdots$$

which suggest $\pi_{2i-2}(F) = 0$.

$\mathrm{BP}\langle n\rangle$ as a $\mathbb{Z}$-equivariant $\mathbb{E}_1 \otimes \mathbb{A}_2$-$\mathrm{MU}_{(p)}{}^{\Psi}$-algebra

Proposition

The underlying $\mathbb{E}_1 \otimes \mathbb{A}_2$-$\mathrm{MU}_{(p)}$*-algebra of* $\mathrm{BP}\langle n\rangle$ *lifts to*

$$\mathrm{BP}\langle n\rangle^{\Psi} \in \mathrm{Alg}_{\mathbb{E}_1 \otimes \mathbb{A}_2}(\mathrm{Mod}(\mathrm{Sp}^{B\mathbb{Z}}; \mathrm{MU}_{(p)}{}^{\Psi})).$$

Proof.

We want to lift $\mathrm{BP}\langle n\rangle$ to a $\mathbb{Z}$-equivariant $(\mathbb{E}_1 \otimes \mathbb{A}_2)$-$\mathrm{MU}_{(p)}$-algebra, so we need to equip the $\mathbb{Z}$-equivariant $\mathbb{E}_1$-$\mathrm{MU}_{(p)}$-algebra we constructed with a unital multiplication that is compatible with the $\mathbb{Z}$-equivariant structure. Using $\mathbb{E}_1$-cell theory, we extend $\triangledown$ from a filtration along the left arrow to the dashed arrow.

$$\begin{tikzcd}
\mathrm{BP}\langle n\rangle^{\Psi} \coprod \mathrm{BP}\langle n\rangle^{\Psi} \arrow[rr, "\triangledown"] \arrow[dr] & & \mathrm{BP}\langle n\rangle^{\Psi} \\
& \mathrm{BP}\langle n\rangle^{\Psi} \otimes_{\mathrm{MU}_{(p)}} \mathrm{BP}\langle n\rangle^{\Psi} \arrow[ur, dashed] &
\end{tikzcd}$$

By calculation, we have $\operatorname{fib} \mathbb{H}(s) = \mathbb{H}(\mathrm{BP}\langle n\rangle) \otimes_{\mathrm{MU}_{(p)}{}^{\Psi}} \mathbb{H}(\mathrm{BP}\langle n\rangle)$ so its homotopy groups are concentrated in positive even degrees.

We can pick $X_j = \Sigma^{2k_i}\,\mathrm{MU}_{(\mathrm{p})}{}^{\Psi}(j)$ for $j < k_i$ and have a filtration

$$\mathrm{BP}\langle \mathrm{n}\rangle^{\Psi} \coprod \mathrm{BP}\langle \mathrm{n}\rangle^{\Psi} = R_0 \to R_1 \to \cdots \to R_\infty \xrightarrow{\cong} \mathrm{BP}\langle \mathrm{n}\rangle^{\Psi} \otimes_{\mathrm{MU}_{(\mathrm{p})}} \mathrm{BP}\langle \mathrm{n}\rangle^{\Psi}$$

Here R_i is obtained by R_{i-1} via the lower pushout square

$$\begin{tikzcd}
\Sigma^{2k_i}\,\mathrm{MU}_{(\mathrm{p})}{}^{\Psi}(j) \arrow[r] \arrow[d] & 0 \arrow[d] \arrow[dddr, dashed, "h_i"] & \\
\mathrm{MU}_{(\mathrm{p})}{}^{\Psi}\{\Sigma^{2k_i}\,\mathrm{MU}_{(\mathrm{p})}{}^{\Psi}(j)\} \arrow[r, "aug"] \arrow[d] & \mathrm{MU}_{(\mathrm{p})}{}^{\Psi} \arrow[d] \arrow[ddr, dashed, "g_i"] & \\
R_{i-1} \arrow[r, "r_i"] \arrow[drr, bend right, "f_{i-1}"'] & R_i \arrow[dr, dashed, "f_i"] & \\
 & & \mathrm{BP}\langle \mathrm{n}\rangle^{\Psi}
\end{tikzcd}$$

Similar to the last proof, the existence of a lift f_i boils is equivalent to the existence of a lift h_i, which means the the composition of maps from $\Sigma^{2k_i} \mathrm{MU_{(p)}}^{\Psi}(j)$ to $\mathrm{BP\langle n\rangle}^{\Psi}$ is nullhomotopic. Note that in general we have a exact sequence

$$\mathrm{Map}_{\mathcal{C}^{B\mathbb{Z}}}(X^{\Psi_X}, Y^{\Psi_Y}) \to \mathrm{Map}_{\mathcal{C}}(X, Y) \xrightarrow{F} \mathrm{Map}_{\mathcal{C}}(X, Y)$$

where $F(f)(x) = \Psi_X(x)f - \Psi_Y f(x)$. This induces a long exact sequence

$$\cdots \to \pi_1(\mathrm{Map}_{\mathcal{C}}(X, Y)) \to \pi_0(\mathrm{Map}_{\mathcal{C}^{B\mathbb{Z}}}(X^{\Psi_X}, Y^{\Psi_Y}) \to \pi_0(\mathrm{Map}_{\mathcal{C}}(X, Y))$$

$$\to \pi_0(\mathrm{Map}_{\mathcal{C}}(X, Y)) \to \cdots$$

Take $\mathcal{C} = \mathrm{Mod}(\mathrm{Sp}^{B\mathbb{Z}}; MU^{\Psi}_{(p)})), X = \Sigma^{2k_i}\,\mathrm{MU_{(p)}}^{\Psi}(j), Y = \mathrm{BP}\langle \mathrm{n}\rangle$. We have

$$\pi_1(\mathrm{Map}_{\mathcal{C}}(X, Y)) = \pi_{2k_i+1}(\mathrm{BP}\langle \mathrm{n}\rangle) = 0$$

so the middle map

$$\pi_0(\mathrm{Map}_{\mathcal{C}^{B\mathbb{Z}}}(X^{\Psi_X}, Y^{\Psi_Y})) \to \pi_0(\mathrm{Map}_{\mathcal{C}}(X, Y))$$

is injective. By the forgetful-tensor up adjunction, an element $f \in \mathrm{Map}_{\mathcal{C}}(X, Y)$ corresponds to $\alpha \in \pi_{2k_i}(\mathrm{BP}\langle \mathrm{n}\rangle)$. Therefore, in this case

$$\begin{aligned}\pi_0(\mathrm{Map}_{\mathcal{C}^{B\mathbb{Z}}}(X^{\Psi_X}, Y^{\Psi_Y}) &= \ker(\pi_0(\mathrm{Map}_{\mathcal{C}}(X, Y)) \xrightarrow{F} \pi_0(\mathrm{Map}_{\mathcal{C}}(X, Y)) \\ &= \ker(\pi_{2k_i}(\mathrm{BP}\langle \mathrm{n}\rangle) \xrightarrow{F} \pi_{2k_i}(\mathrm{BP}\langle \mathrm{n}\rangle)).\end{aligned}$$

We know Ψ acts on $\pi_{2k_i}(\mathrm{MU_{(p)}}(j))$ by multiplying by l^{k_i+j} and Ψ acts on $\pi_{2k_i}(\mathrm{BP}\langle \mathrm{n}\rangle)$ by l^{k_i}. Since $j \neq k_i$, the image of F here is never 0, so $\mathrm{Map}(\Sigma^{2k_i}\,\mathrm{MU_{(p)}}^{\Psi}(j), \mathrm{BP}\langle \mathrm{n}\rangle) = 0$.

Locally unipotent action

Proposition (Corollary A.27, Appendix A)

Let $\mathcal{C}$ be a presentable stable category with a compact generator V. An $X \in \mathcal{C}^{B\mathbb{Z}}$ is locally unipotent if and only if the $\mathbb{Z}$-action on the homotopy groups $\pi_0(\mathrm{Map}_{\mathcal{C}}(\Sigma^m V, X))$ is locally unipotent for all m.

Corollary

The constructed Adams operation on $\mathrm{BP}\langle \mathrm{n}\rangle$ *is locally unipotent in p-compelete spectra after p-compeletion.*

Proof.

We choose the Moore spectrum $\mathbb{S}/p$ to be the compact generators in the category of p-compelete spectra and get a long exact sequence of homotopy group

$$\cdots \pi_{m+1}X \xrightarrow{p} \pi_{m+1}X \to \pi_0(\Sigma^m \mathbb{S}/p, X) \to \pi_m X \xrightarrow{p} \pi_m X \to \cdots$$

Taking $X = \mathrm{BP}\langle \mathrm{n}\rangle$, and we have $\pi_0(\Sigma^m \mathbb{S}/p, \mathrm{BP}\langle \mathrm{n}\rangle) = \pi_{m+1}\mathrm{BP}\langle \mathrm{n}\rangle/(p)$. We know the homotopy groups of $\mathrm{BP}\langle \mathrm{n}\rangle$ concentrate on degrees that are multiple of $2p-2$, so Ψ^l acts by multiplying by $l^{k(p-1)}$ on $\pi_{2k(p-1)}$. By Fermat's theorem, the action is identity mod p.

Putting the pieces together...

Proof. Note that $\mathbb{Z}_p^\times$ lies in the center of $\mathbb{G}_n$, so Ψ^l commutes with the action of $\mu^{p^n-1} \rtimes \mathrm{Gal}(\mathbb{F}_p)$. Therefore, there is an identification in $\mathrm{Alg}_{\mathbb{E}_1}(\mathrm{Sp}^{B\mathbb{Z}})$:

$$\begin{tikzcd}
L_{T(n)}\,\mathrm{BP}\langle \mathrm{n} \rangle^{\Psi} \arrow[r, "\iota"] \arrow[d, "\cong"'] & E_n^{\Psi} \arrow[d, "\cong"] \\
(E_n^{\Psi})^{h\mu_{p^n-1} \rtimes \mathrm{Gal}(\mathbb{F}_p)} \arrow[r] & E_n^{\Psi}
\end{tikzcd}$$

Reference

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