Constructing an Adams Operation on $BP\langle n \rangle$

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Introduction

- Objects: $MU_{(p)}$, E_n , $BP\langle n \rangle$
- Ingredients: Adams operation on $MU_{(p)}$ (stable Adams conjecture), Adams operation on E_n (Hopkins–Miller)
- Goal: Constructing a locally unipotent Adams operation on $BP\langle n \rangle$ that is compatible with the Adams operations above

Theorem (5.4, Burkland–Hahn–Levy–Schlank)

The $(\mathbb{E}_1 \otimes \mathbb{A}_2)$ - $MU_{(p)}$ -algebra underlying the \mathbb{E}_3 - $MU_{(p)}$ -algebra $\mathsf{BP}\langle \mathsf{n} \rangle$ admits a lift

$$\mathsf{BP}\langle \mathsf{n}\rangle^{\Psi}\in\mathsf{Alg}_{\mathbb{E}_1\otimes\mathbb{A}_2}(\mathsf{Mod}(\mathsf{Sp}^B\mathbb{Z};\mathit{MU}^{\Psi}_{(p)}))$$

such that

• there is a map $\iota : \mathsf{BP}\langle \mathsf{n} \rangle^{\Psi} \to E_n^{\Psi}$ in $\mathsf{Alg}_{\mathbb{E}_1}(\mathsf{Mod}(\mathsf{Sp}^B\mathbb{Z}; MU_{(n)}^{\Psi})).$

Theorem (BHLS (cont.))

• an identification in $Alg_{\mathbb{E}_1}(Sp^{B\mathbb{Z}})$:

$$\begin{array}{ccc} L_{\mathcal{T}(n)} \operatorname{\mathsf{BP}} \langle \mathsf{n} \rangle^{\Psi} & \xrightarrow{\iota} & E_n^{\Psi} \\ & & & \downarrow \cong \\ & & & \downarrow \cong \\ (E_n^{\Psi})^{h\mu_{p^n-1} \rtimes \mathsf{Gal}(\mathbb{F}_p)} & \longrightarrow & E_n^{\Psi} \end{array}$$

where $\mu_{p^n-1} \rtimes \mathsf{Gal}(\mathbb{F}_p)$ fits into

$$\mu_{p^n-1} \xrightarrow{} \mu_{p^n-1} \rtimes \operatorname{Gal}(\mathbb{F}_p) \xrightarrow{} \operatorname{Gal}(\mathbb{F}_p)$$

$$\downarrow^{\cong}$$

$$\mathbb{S}_n \xrightarrow{} \mathbb{G}_n \xrightarrow{} \operatorname{Gal}(\mathbb{F}_p)$$

• the underlying \mathbb{Z} -action on $BP\langle n \rangle$ is locally unipotent in p-complete spectra after p-completion.

Outline of the construction

• Refine the \mathbb{E}_3 -algebra structure on E_n to an \mathbb{E}_3 -MU_(p)-algebra together with a \mathbb{E}_3 -MU_(p)-algebra map:

$$\iota:\mathsf{BP}\langle\mathsf{n}\rangle\to E_n.$$

(Theorem 5.9, Theorem 5.10)

- Use the self-centrality of E_n and universal property of center to show that the \mathbb{E}_3 -MU_(p)-algebra structure on E_n corresponds to an \mathbb{E}_4 -algebra map MU_(p) $\to E_n$. (Theorem 5.12, 5.13)
- Base case: Show that the underlying \mathbb{E}_3 -algebra map of the \mathbb{E}_4 -algebra map $\mathsf{MU}_{(p)} \to E_n$ can be refined to a \mathbb{Z} -equivariant \mathbb{E}_3 -algebra map $\mathsf{MU}_{(p)} \to E_n$, so E_n^{Ψ} lifts to a \mathbb{Z} -equivariant \mathbb{E}_2 - $\mathsf{MU}_{(p)}^{\Psi}$ -algebra. (Theorem 5.16)

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• Induction: Refine the \mathbb{E}_1 -MU_(p)-algebra of BP $\langle n \rangle$ to

$$\mathsf{BP}\langle \mathsf{n} \rangle^{\Psi} \in \mathsf{Alg}_{\mathbb{E}_1}(\mathsf{Mod}(\mathsf{Sp}^{B\,\mathbb{Z}};\mathsf{MU}_{(\mathsf{p})}{}^{\Psi})).$$

and extend $\iota: \mathsf{BP}\langle \mathsf{n} \rangle \to E_n$ to a \mathbb{Z} -equivariant map between $\mathbb{E}_1\text{-MU}_{(\mathsf{p})}^{\Psi}$ -algebras. (Theorem 5.29)

 \bullet Lift the underlying $\mathbb{E}_1 \otimes \mathbb{A}_2\text{-MU}_{(p)}\text{-algebra of BP}\langle \mathsf{n} \rangle$ to

$$\mathsf{BP}\langle \mathsf{n}
angle^{\Psi} \in \mathsf{Alg}_{\mathbb{E}_1 \, \otimes \, \mathbb{A}_2}(\mathsf{Mod}(\mathsf{Sp}^B{}^{\mathbb{Z}}; \mathsf{MU}_{(\mathsf{p})}{}^{\Psi})).$$

(Theorem 5.33)

• Use the fact that $\Phi^l \in \mathbb{G}_n$ is central to show that the Adams operation and taking the homotopy fixed point commute + prove local indempotence.

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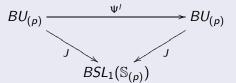
Adams operation on $MU_{(p)}$ and E_n

Construction (Adams operation on E_n)

Let l=3 for p=2 and l=2 for odd p for this talk. Let $\Psi^l \in \mathbb{G}_n : E_n \to E_n$ that arises from the \mathbb{Z}_p^{\times} -action on the chosen formal group of height n over $\overline{\mathbb{F}_p}$, such that it acts on $\pi_{2k}(E_n)$ by l^k .

Construction (Adams operation on $MU_{(p)}$)

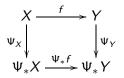
The stable Adams conjecture gives us



Taking the Thom spectrum, we get an Admas operation on $MU_{(p)}$.

Notation

• The category $\mathsf{Mod}(\mathsf{MU}_{(\mathsf{p})}^{\Psi};\mathsf{Sp}^{B\mathbb{Z}})$ has objects $\{(X,\Psi_X):X\in\mathsf{Sp}^{B\mathbb{Z}}\cap\mathsf{Mod}(MU),\mathsf{MU}_{(\mathsf{p})}\text{-module map}\ \Psi_X:X\to\Psi_*(X)\}$ and morphisms $f:X\to Y$ are commutative diagrams of $\mathsf{MU}_{(\mathsf{p})}\text{-module maps}$



- $\mathbb{S}_{(p)}(j)$: invertible object in *p*-complete spectra where Ψ^I acts by multiplying by I^j on $\pi_0(\mathbb{S}_{(p)})^{\times}$.
- $MU_{(p)}(j) = MU_{(p)} \otimes_{\mathbb{S}_p} \mathbb{S}_p(j)$.
- $S^{n,[l]}$: sphere with a degree-l self map.

\mathbb{E}_3 -MU_(p)-algebra structure on E_n

We have a classical result that there is an isomorphism

$$L_{\mathcal{T}(n)} \operatorname{\mathsf{BP}} \langle \mathsf{n} \rangle \xrightarrow{\cong} (E_n)^{h\mu_{p^n-1} \rtimes \operatorname{\mathsf{Gal}}(\mathbb{F}_p)},$$

and our goal here is to define a $MU_{(p)}$ -module structure on E_n . In the construction of $BP\langle n \rangle$, we define $MU_{(p)}[y]$ as the Thom spectrum of

$$\mathbb{N} \xrightarrow{p^n-1} \mathbb{Z} \to \mathrm{Pic}(\mathsf{MU}_{(p)}).$$

Analogously, we define $MU_{(p)}[y^{1/(p^n-1)}]$ as as the Thom spectrum of

$$\mathbb{N} \xrightarrow{1} \mathbb{Z} \to \mathrm{Pic}(\mathsf{MU}_{(p)}).$$

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We define

$$\mathsf{BP}\langle \mathsf{n}\rangle[v_n^{1/(p^n-1)}] := \mathsf{MU}_{(\mathsf{p})}[y^{1/(p^n-1)}] \otimes_{\mathsf{MU}_{(\mathsf{p})}} \mathsf{BP}\langle \mathsf{n}\rangle$$

and we obtained a \mathbb{E}_3 -MU_(p)[$y^{1/(p^n-1)}$]-algebra.

Lemma

The commutative algebra map

$$\mathbb{W}(\mathbb{F}_p) \otimes \mathsf{MU}_{(p)}[y^{\pm 1}] \to \mathsf{colim}_k \, \mathbb{W}(\mathbb{F}_{p^k}) \otimes \mathsf{MU}_{(p)}[y^{\pm 1/(p^n - 1)}]$$

is a $\mu_{p^n-1} \rtimes Gal(\mathbb{F}_p)$ pro-Galois extension.

Proof. (sketch)

Lifting the $\mu_{p^n-1} \rtimes Gal(\mathbb{F}_p)$ pro-Galois extension on homotopy groups.

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Proposition

There is an identification of underlying \mathbb{E}_3 -algebra

$$L_{T(n)} \operatorname{BP}\langle \mathsf{n} \rangle \longrightarrow L_{T(n)}(\operatorname{colim}_k \mathbb{W}(\mathbb{F}_{p^k}) \otimes \operatorname{BP}\langle \mathsf{n} \rangle [v_n^{1/(p^n-1)}])$$

$$\cong \bigvee_{\substack{\subseteq \\ (E_n)^{h\mu_{p^n-1}} \rtimes \operatorname{Gal}(\mathbb{F}_p)}} \bigvee_{\substack{\subseteq \\ E_n}} E_n$$

Proof. (sketch)

The homotopy group of $L_{\mathcal{T}(n)}(\operatorname{colim}_k \mathbb{W}(\mathbb{F}_{p^k}) \otimes \operatorname{BP}\langle n \rangle [v_n^{1/(p^n-1)}])$ agrees with a Lubin–Tate theory, so it agrees as an \mathbb{E}_3 -algebra with a Lubin–Tate theory, which is E_n as there is one formal group over an algebraically closed field up to isomorphism.

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Therefore, we obtain a refinement of the underlying \mathbb{E}_3 -algebra structure on E_n to an \mathbb{E}_3 -MU_(p)-algebra structure and a map of \mathbb{E}_3 -MU_(p)-algebra:

$$\iota: \mathsf{BP}\langle \mathsf{n} \rangle \to E_n$$
.

The \mathbb{Z} -equivariant \mathbb{E}_2 -MU_(p)-algebra structure on E_n

It's a fact [Lur17, Section 5.3] that for for an \mathbb{E}_{m} -algebra R, its center $\mathcal{Z}_{\mathbb{E}_{m}}(R)$ is the terminal \mathbb{E}_{m+1} -algebra A such that R lifts to a \mathbb{E}_{m} -A-algebra. Explicitly, we have that R lifts to a \mathbb{E}_{m} - $\mathcal{Z}_{\mathbb{E}_{m}}$ -algebra, and for any \mathbb{E}_{m+1} -algebra B such that R lifts to a \mathbb{E}_{m} -B-algebra, there is a corresponding \mathbb{E}_{m+1} -algebra map $B \to \mathcal{Z}_{\mathbb{E}_{m}}R$.

Proposition

The \mathbb{E}_m center of E_n is isomorphic to E_n for $m \geq 2$.

Therefore, the structure map of \mathbb{E}_3 -MU_(p)-algebra:

$$f: \mathsf{MU}_{(p)} \to E_n$$

corrsponds to a map of \mathbb{E}_4 -algebra:

$$\bar{f}: \mathsf{MU}_{(\mathsf{p})} \to \mathcal{Z}_{\mathbb{E}_3} E_n \cong E_n.$$

Our goal is to refine E_n^{Ψ} to a \mathbb{Z} -equivariant \mathbb{E}_2 -MU_(p) $^{\Psi}$ -algebra from the \mathbb{E}_3 -MU_(p)-algebra structure on E_n , so it suffices to refine the \mathbb{E}_4 -algebra map $\bar{\iota}$ to a \mathbb{Z} -equivariant \mathbb{E}_3 -algebra map

$$\mathsf{MU_{(p)}}^\Psi \to E_n^\Psi.$$

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Proposition

The underlying \mathbb{E}_3 -algebra map of any \mathbb{E}_4 -algebra map \bar{f} can be refined to a \mathbb{Z} -equivariant \mathbb{E}_3 -algebra map

$$\mathsf{MU_{(p)}}^{\Psi} \to E_n^{\Psi}$$
.

Proof. (sketch)

By [ACB19, Theorem 3.5], the space of \mathbb{E}_4 -algebra map \bar{f} is isomorphic to the space of nullhomotopies of the composite

$$BU(p) \stackrel{J}{\to} BSL(\mathbb{S}_{(p)}) \to BSL_1(E_n)$$

in the category of 4-fold loop maps, which is then equivalent to this space of nullhomotopies in pointed spaces of the composite

$$B^4BU(p) \stackrel{J}{\to} B^5SL(\mathbb{S}_{(p)}) \to B^5SL_1(E_n).$$

We prove this theorem by showing that every nullhomotopies of this composite can be refined to a \mathbb{Z} -equivariant nullhomotopy of the composite

$$B^{4,[I^2]}BU(p) \stackrel{J}{ o} B^{5,[I^2]}SL(\mathbb{S}_{(p)}) o B^{5,[I^2]}SL_1(E_n).$$

Applying $\Omega^{4,[l^2]}$ to the sequence above, we get a \mathbb{Z} -equivariant nullhomotopy of the composite in the category of 3-fold loop maps as $S^{4,[l^2]} = S^{3,[0]} \wedge S^{1,[l^2]}$.

Corollary

The \mathbb{E}_3 -MU_(p)-algebra structure on E_n can be used to refine E_n^{Ψ} to a \mathbb{Z} -equivariant \mathbb{E}_2 -MU_(p) $^{\Psi}$ -algebra.

\mathbb{E}_1 -Cell Theory

Definition (Connective algebra)

For a stable, presentably symmeytric moinodal category \mathcal{C} , an algebra object $A \in \mathsf{Alg}(\mathcal{C})$ is called connected if A is connective and $\pi_0(A) = \pi_0(\mathbbm{1})$. Let $\mathsf{Alg}(\mathcal{C})^{\geq 1}$ denote the connected algebra object in \mathcal{C} .

Definition (Functor ℍ)

We define $\ensuremath{\mathbb{H}}$ to be the composite of

$$\mathsf{Alg}(\mathcal{C})^{\geq 1} \xrightarrow{\otimes \pi_0 \, \mathbb{1}} \mathsf{Alg}(\mathsf{Mod}(\mathcal{C}; \pi_0(\mathbb{1}))) \xrightarrow{\Sigma^{\infty}} \mathsf{Mod}(\mathcal{C}; \pi_0(\mathbb{1})).$$

Concretely, we have

$$\mathbb{H}(A) = \mathrm{fib}(A \otimes \pi_0(\mathbb{1}) \to \pi_0(\mathbb{1}) \otimes_{\pi_0(A)} \pi_0(\mathbb{1})).$$

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Example

- $\mathbb{H}(\mathbb{1}\{X\}) \cong X$.
- Take $\mathcal{C} = \mathsf{Mod}(\mathsf{MU}_{(p)})$, then $\pi_0(\mathbbm{1}) = \pi_0(\mathsf{MU}_{(p)}) = \mathbbm{Z}_{(p)}$. It is calculated that $\pi_*(\mathbbm{H}(\mathsf{BP}\langle\mathsf{n}\rangle))$ is torsion-free, finitely generated, and concentrated in positive odd degrees.
- $\operatorname{fib}(\mathbb{H}(A \mid B) \to \mathbb{H}(A \otimes B)) \cong \mathbb{H}(A) \otimes_{\pi_0(B)} \mathbb{H}(B).$

Lemma (\mathbb{E}_1 -celullar approximation)

Let $f: A \to B$ be a map in $Alg(\mathcal{C})^{\geq 1}$ and let

$$\mathbb{H}(A) = M_0 \xrightarrow{m_1} M_1 \xrightarrow{m_1} M_2 \to \cdots \to M_{\infty} = \mathbb{H}(B)$$

be an increasing filtration such that

- $\mathbf{0}$ m; is k_i -connnective for a non-decreasing sequence k_i with $k_1 \geq 0$,
- **2** for each $i \geq 1$, there are objects $X_i \in \mathcal{C}_{\geq k}$ such that
 - $X_i \otimes \pi_0(\mathbb{1}) \cong \mathrm{fib}(m_i)$ and $[X_i, Y] = 0$ for any $(k_i + 2)$ -connective Y.

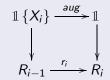
Lemma (\mathbb{E}_1 -celullar approximation, cont.)

Then there exists a filtration in $Alg(\mathcal{C})^{\geq 1}$:

$$A:=R_0\xrightarrow{r_1}R_1\xrightarrow{r_2}\to\cdots\to R_\infty\to B$$

such that

- **1** the map $R_i \rightarrow B$ is k_{i+1} -connective,
- ② the functor \mathbb{H} sends $\{R_i\}$ to $\{M_i\}$,
- each R_i is built as a pushout



4 the map $R_{\infty} \to B$ is ∞ -connective.

We apply this lemma to the category of $MU_{(p)}$ -module spectra. Using above examples, we will see shortly that it's relatively easy to construct a filtration on the homology level $\mathbb H$ (as a split Posnikov tower where we know what the fibers are) that satisfies the connective and coconnective hypothesis. We want to obtain a filtration on the spectrum level, which allows us to extend the $\mathbb Z$ -equivariant structure level by level from $MU_{(p)}$ to $BP\langle n \rangle$ via taking pushout.

$\mathsf{BP}\langle\mathsf{n}\rangle$ as a \mathbb{Z} -equivariant $\mathbb{E}_1\text{-}\mathsf{MU}_{(p)}^{\Psi}$ -algebra

Proposition

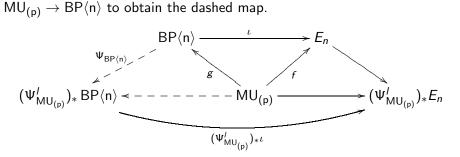
The underlying \mathbb{E}_1 -MU_(p)-algebra of BP $\langle n \rangle$ lifts to

$$\mathsf{BP}\langle \mathsf{n} \rangle^{\Psi} \in \mathsf{Alg}(\mathsf{Mod}(\mathsf{Sp}^B\mathbb{Z}; \mathsf{MU}_{(\mathsf{p})}^{\Psi})).$$

and $\iota:\mathsf{BP}\langle\mathsf{n}\rangle\to E_n$ respects the \mathbb{Z} -equivariant \mathbb{E}_1 -MU_(p)-algebra structure.

Proof.

Using \mathbb{E}_1 -cell theory, we extend the structure map $MU_{(p)} \to (\Psi_{MU_{(p)}}^I *) BP\langle n \rangle$ by a filtration along the structure map



We have a filtration

$$\mathbb{H}(\mathsf{MU}_{(\mathsf{p})}) = 0 \to \Sigma^{1}(\pi_{1}(\mathbb{H}(\mathsf{BP}\langle\mathsf{n}\rangle)) \to \Sigma^{1}(\pi_{1}(\mathbb{H}(\mathsf{BP}\langle\mathsf{n}\rangle)) \oplus \Sigma^{3}(\pi_{3}(\mathbb{H}(\mathsf{BP}\langle\mathsf{n}\rangle))))$$

$$\bigoplus \Sigma^{2i-1}(\pi_{2i-1}(\mathbb{H}(\mathsf{BP}\langle\mathsf{n}\rangle)) \cong \mathbb{H}(\mathsf{BP}\langle\mathsf{n}\rangle)$$

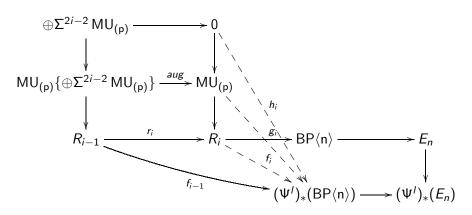
where each fiber is $\pi_{2i-1}(\mathbb{H}(\mathsf{BP}(\mathsf{n})))$ which, by the examples above, is a finite sum of copies of $\mathbb{Z}_{(p)}$.

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We take X_i to be a sum of copies of $\Sigma^{2i-2} MU_{(p)}$, so we have a filtration

$$\mathsf{MU}_{(p)} = R_0 \xrightarrow{r_1} R_1 \xrightarrow{r_2} \to \cdots \to R_\infty \xrightarrow{\cong} \mathsf{BP}\langle \mathsf{n} \rangle$$

as \mathbb{E}_1 -MU_(p)-algebra. Here R_i is obtained by R_{i-1} via the lower pushout square



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Given the commutative diagram with all solid arrows, we want to find a lift f_i . By the universal property of pushout, it suffices to find a lift g_i . Again, by the universal property of free object, it boils down to find a lift h_i . After simplifying the diagram, we see that we need to find a dashed lift, which in this case amounts to find a nullhomotopy of $\oplus \Sigma^{2i-2} MU_{(p)} \to hfib((\Psi')_*\iota).$

$$\begin{array}{c} \operatorname{hfib}((\Psi^I)_*\iota) =: F \\ \downarrow \\ \oplus \Sigma^{2i-2} \operatorname{MU}_{(p)} \xrightarrow{h_i} (\Psi^I)_*(BP\langle n \rangle) \\ \downarrow \\ 0 \xrightarrow{h_i} (\Psi^I)_*(E_n) \end{array}$$

There exists a long exact sequence of homotopy group:

$$\cdots \to \pi_{2i-1}(E_n) \to \pi_{2i-2}(F) \to \pi_{2i-2}(\mathsf{BP}(\mathsf{n})) \hookrightarrow \pi_{2i-2}(E_n) \to \cdots$$

which suggest $\pi_{2i-2}(F) = 0$.

 $\mathsf{BP}\langle\mathsf{n}\rangle$ as a \mathbb{Z} -equivariant $\mathbb{E}_1\otimes\mathbb{A}_2\text{-}\mathsf{MU_{(p)}}^\Psi$ -algebra

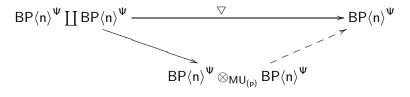
Proposition

The underlying $\mathbb{E}_1 \otimes \mathbb{A}_2\text{-MU}_{(p)}\text{-algebra of BP}\langle n \rangle$ lifts to

$$\mathsf{BP}(\mathsf{n})^{\Psi} \in \mathsf{Alg}_{\mathbb{E}_1 \otimes \mathbb{A}_2}(\mathsf{Mod}(\mathsf{Sp}^B\mathbb{Z}; \mathsf{MU}_{(\mathsf{p})}^{\Psi})).$$

Proof.

We want to lift BP $\langle n \rangle$ to a \mathbb{Z} -equivariant $(\mathbb{E}_1 \otimes \mathbb{A}_2)$ -MU $_{(p)}$ -algebra, so we need to equip the \mathbb{Z} -equivariant \mathbb{E}_1 -MU $_{(p)}$ -algebra we constructed with a unital multiplication that is compatible with the \mathbb{Z} -equivariant structure. Using \mathbb{E}_1 -cell theory, we extend ∇ from a filtration along the left arrow to the dashed arrow.



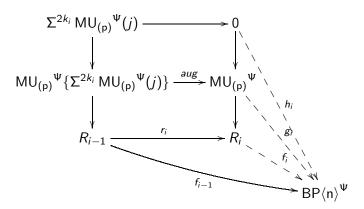
By calculation, we have $\operatorname{fib}\mathbb{H}(s)=\mathbb{H}(\mathsf{BP}\langle\mathsf{n}\rangle)\otimes_{\mathsf{MU}_{(p)}}{}^{\Psi}\mathbb{H}(\mathsf{BP}\langle\mathsf{n}\rangle)$ so its homotopy groups are concentrated in positive even degrees.

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We can pick $X_j = \Sigma^{2k_i} MU_{(p)}^{\Psi}(j)$ for $j < k_i$ and have a filtration

$$\mathsf{BP}\langle \mathsf{n}\rangle^{\Psi}\coprod \mathsf{BP}\langle \mathsf{n}\rangle^{\Psi} = \mathit{R}_0 \to \mathit{R}_1 \to \cdots \to \mathit{R}_{\infty} \xrightarrow{\cong} \mathsf{BP}\langle \mathsf{n}\rangle^{\Psi} \otimes_{\mathsf{MU}_{(p)}} \mathsf{BP}\langle \mathsf{n}\rangle^{\Psi}$$

Here R_i is obtained by R_{i-1} via the lower pushout square



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Similar to the last proof, the existence of a lift f_i boils is equivalent to the existence of a lift h_i , which means the the composition of maps from $\Sigma^{2k_i} \, \mathsf{MU_{(p)}}^{\Psi}(j)$ to $\mathsf{BP}\langle \mathsf{n} \rangle^{\Psi}$ is nullhomotopic. Note that in general we have a exact sequence

$$\mathsf{Map}_{\mathcal{C}^{B\mathbb{Z}}}(X^{\Psi_X},Y^{\Psi_Y}) \to \mathsf{Map}_{\mathcal{C}}(X,Y) \xrightarrow{F} \mathsf{Map}_{\mathcal{C}}(X,Y)$$
 where $F(f)(x) = \Psi_X(x)f - \Psi_Y f(x)$. This induces a long exact sequence
$$\cdots \to \pi_1(\mathsf{Map}_{\mathcal{C}}(X,Y)) \to \pi_0(\mathsf{Map}_{\mathcal{C}^B\mathbb{Z}}(X^{\Psi_X},Y^{\Psi_Y}) \to \pi_0(\mathsf{Map}_{\mathcal{C}}(X,Y)) \to \cdots \to \pi_0(\mathsf{Map}_{\mathcal{C}}(X,Y)) \to \cdots$$

Take $C = \mathsf{Mod}(\mathsf{Sp}^{B\mathbb{Z}}; MU^{\Psi}_{(p)})), X = \Sigma^{2k_i} \mathsf{MU}_{(p)}^{\Psi}(j), Y = \mathsf{BP}\langle \mathsf{n} \rangle.$ We have

$$\pi_1(\mathsf{Map}_\mathcal{C}(X,Y)) = \pi_{2k_i+1}(\mathsf{BP}\langle\mathsf{n}\rangle) = 0$$

so the middle map

$$\pi_0(\mathsf{Map}_{\mathcal{C}^{B\,\mathbb{Z}}}(X^{\Psi_X},Y^{\Psi_Y})) o\pi_0(\mathsf{Map}_{\mathcal{C}}(X,Y))$$

is injective. By the forgetful-tensor up adjunction, an element $f \in \mathsf{Map}_{\mathcal{C}}(X,Y)$ corresponds to $\alpha \in \pi_{2k_i}(\mathsf{BP}\langle\mathsf{n}\rangle)$. Therefore, in this case

$$\pi_0(\mathsf{Map}_{\mathcal{C}^{B\mathbb{Z}}}(X^{\Psi_X},Y^{\Psi_Y}) = \ker(\pi_0(\mathsf{Map}_{\mathcal{C}}(X,Y)) \xrightarrow{F} \pi_0(\mathsf{Map}_{\mathcal{C}}(X,Y))$$
$$= \ker(\pi_{2k_i}(\mathsf{BP}\langle\mathsf{n}\rangle) \xrightarrow{F} \pi_{2k_i}(\mathsf{BP}\langle\mathsf{n}\rangle)).$$

We know Ψ acts on $\pi_{2k_i}(\mathsf{MU}_{(p)}(j))$ by multiplying by I^{k_i+j} and Ψ acts on $\pi_{2k_i}(\mathsf{BP}\langle\mathsf{n}\rangle)$ by I^{k_i} . Since $j\neq k_i$, the image of F here is never 0, so $\mathsf{Map}(\Sigma^{2k_i}\mathsf{MU}_{(p)}\Psi(j),\mathsf{BP}\langle\mathsf{n}\rangle)=0$.

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Locally unipotent action

Proposition (Corollary A.27, Appendix A)

Let \mathcal{C} be a presentable stable category with a compact generator V. An $X \in \mathcal{C}^{B\mathbb{Z}}$ is locally unipotent if and only if the \mathbb{Z} -action on the homotopy groups $\pi_0(\mathsf{Map}_{\mathcal{C}}(\Sigma^m V, X))$ is locally unipotent for all m.

Corollary

The constructed Adams operation on $BP\langle n \rangle$ is locally unipotent in p-compelete spectra after p-compeletion.

Proof.

We choose the Moore spectrum \mathbb{S}/p to be the compact generators in the category of p-compelete spectra and get a long exact sequence of homotopy group

$$\cdots \pi_{m+1}X \xrightarrow{p} \pi_{m+1}X \to \pi_0(\Sigma^m \mathbb{S}/p, X) \to \pi_mX \xrightarrow{p} \pi_mX \to \cdots$$

Taking $X = \mathsf{BP}\langle \mathsf{n} \rangle$, and we have $\pi_0(\Sigma^m \mathbb{S}/p, \mathsf{BP}\langle \mathsf{n} \rangle) = \pi_{m+1} \, \mathsf{BP}\langle \mathsf{n} \rangle / (p)$. We know the homotopy groups of $\mathsf{BP}\langle \mathsf{n} \rangle$ concentrate on degrees that are multiple of 2p-2, so Ψ^I acts by multiplying by $I^{k(p-1)}$ on $\pi_{2k(p-1)}$. By Fermat's theorem, the action is identity mod p.

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Putting the pieces together...

Proof. Note that \mathbb{Z}_p^{\times} lies in the center of \mathbb{G}_n , so Ψ^I commutes with the action of $\mu^{p^n-1} \rtimes \text{Gal}(\mathbb{F}_p)$. Therefore, there is an identification in $\text{Alg}_{\mathbb{E}_1}(\text{Sp}^{B\mathbb{Z}})$:

$$\begin{array}{ccc} L_{\mathcal{T}(n)} \operatorname{\mathsf{BP}} \langle \mathsf{n} \rangle^{\Psi} & \xrightarrow{\iota} & E_n^{\Psi} \\ & & & \downarrow \cong \\ (E_n^{\Psi})^{h\mu_{p^n-1} \rtimes \operatorname{\mathsf{Gal}}(\mathbb{F}_p)} & \longrightarrow E_n^{\Psi} \end{array}$$

Reference

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